Fermionizing a small gas of ultracold bosons

B. Paredes,[1] P. Zoller,[2] and J. I. Cirac[1]
[1] *Max–Planck Institute for Quantum Optics, Garching, Germany*
[2] *Institute for Theoretical Physics, University of Innsbruck, Austria*
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We study the physics of a rapidly–rotating gas of ultracold atomic bosons, with an internal degree of freedom. We show that in the limit of rapid rotation of the trap the problem exactly maps onto that of non–interacting fermions with spin in the lowest Landau level. The spectrum of the real bosonic system is identical to the one of the effective fermions, with the same eigenvalues and the same density of states. When the ratio of the number of atoms to the spin degeneracy is an integer number, the ground state for the effective fermions is an integer quantum Hall state. The corresponding bosonic state is a fractional quantum Hall liquid whose filling factor ranges in the sequence $\nu = 1/2, 2/3, 3/4, \ldots$, as the spin degeneracy increases. Anyons with $1/2, 1/3, 1/4, \ldots$ statistics can be created by inserting lasers with the appropriate polarizations. A special situation appears when the spin degeneracy equals the number of atoms in the gas. The ground state is then the product of a completely antisymmetric spin state and a $\nu = 1$ Laughlin state. In this case the system exhibits fermionic excitations with fermionic statistics though the real components are bosonic atoms.

I. INTRODUCTION

Ultra cold atomic gases have been shown to be ideal candidates to observe several intrinsically quantum phenomena, since they can be easily controlled and manipulated by electromagnetic fields. What type of quantum phenomena we can observe depends dramatically on the strength of the atomic interaction, going from one particle phenomena to highly entangled quantum states.

The first experimental highlight with these systems was the achievement of Bose-Einstein condensation of weakly interacting atomic gases [1]. This success motivated a lot of experimental and theoretical work with cold bosons in the limit of weak interactions. In this regime atoms essentially occupy the same single particle state, and the physics of the system is perfectly understood via a Gross-Pitaievski equation for the macroscopic wave function. Many interesting phenomena have been studied in this regime and there still exist certain open questions [2].

On the other hand, it seems clear that going in the direction of strongly correlated atoms, and exploring what happens beyond the validity of mean field theory, promises to open a new path to the observation of novel physical phenomena. Strong correlations between particles have been shown to lead to unusual fascinating effects in condensed matter systems. In the same way, if an atomic system is driven into the strongly interacting regime, similar interesting phenomena may be observed. Moreover, since atomic gases can be well controlled and manipulated, they may provide us with new scenarios for the observation of these phenomena and, perhaps, new ones that have not been accessible so far in solid state environments. The difficulty here is, however, how to drive an atomic system into a strongly interacting regime. In order to enter this regime one should make the typical interaction energy larger than any other (single particle) energy scale in the problem. Naively speaking, it may seem that the easiest way to do this would be just to increase the interaction strength, either by increasing the scattering length or the atomic density. However, though the scattering length can be effectively enlarged by using Feshbach resonances [3], and the atoms squeezed to increase the density, both methods may lead to undesirable instabilities in the system.

Alternatively, in order to make interactions to dominate, one can decrease the single particle energies by creating degeneracies. Here, one possibility is to use internal levels, as in the recent proposal to entangle atomic beams by generation of spin-squeezed states [4]. Degeneracies can also be induced by using a periodic external potential, since single particle states corresponding to different wells will have the same energy. In the limit in which the amplitude of tunneling between wells is small compared to the typical interaction energy, the system has been predicted to undergo a transition from a superfluid phase to a Mott insulator phase [5], which is highly correlated. This quantum phase transition has been recently achieved experimentally in an optical lattice [6]. Another way of achieving a quasi-degeneracy in the atomic motional states is to rotate the trap that confines the atoms. When the frequency of rotation is small compared to the frequency of the trap, the system forms a vortex condensate, with a number of vortices that increases as the rotation becomes faster [7]. This vortex condensate is a non–correlated state, in which all the atoms occupy the same single particle quantum state. However, when the rotation frequency is close to the harmonic trap frequency the system has been predicted to enter a *fractional quantum Hall regime*, in which the atoms are strongly correlated [8, 9]. The ground state is a $\frac{1}{2}$–Laughlin state, a highly entangled liquid with nearly uniform density. This atomic liquid offers the fascinating possibility of creating and manipulating anyons with $\frac{1}{2}$–fractional–statistics [9]. By piercing the system with off resonant localized lasers, $\frac{1}{2}$–Laughlin quasiholes may

be created and manipulated to probe directly their fractional statistics. For the fractional quantum Hall regime to be experimentally achieved in current rotating traps one would need bosonic gases consisting of a small number of atoms, a limit that has remained practically unexplored. A particularly interesting exception is a recent experiment with an optical lattice [6], that has achieved a regime in which a few number of atoms are confined in each well of the lattice. Since the two-dimensional rotating realization of this lattice may be accessible in near future experiments, further theoretical investigation in the physics of small rotating bosonic gases is highly motivated.

In this paper we extend the results of our previous work [9] on a rapidly rotating gas of bosonic atoms, to the case in which the atoms have an internal degree of freedom. We will show that the addition of internal levels introduces new aspects of the problem, and novel exotic correlated phases come out.

The problem of strongly interacting bosons with spin in a rapidly rotating trap is, in principle, very difficult to tackle; in particular it is not easy to guess what type of ground states and excitations appear. In this work we present a formalism, what we call *fermionization* of the bosonic system, within which the problem takes a very simple form. We will show that in the limit of rapid rotation and small temperatures, the problem of interacting bosons maps exactly onto a problem of non-interacting fermions. The ground state corresponds to a Fermi sea in which the fermions fill up the single particle states in order of increasing energy. Furthermore, the lowest energy excitations consist of fermion-hole excitations in the vicinity of the Fermi level. The spectrum of the real bosonic system is identical to the one of the effective non-interacting fermions, with the same eigenvalues and the same density of states.

The fermionization scheme is valid for any number of particles N and any number of internal states n, provided that the atomic interaction does not depend on the internal states of the atoms. Perhaps the most interesting states appear whenever the ratio N/n is an integer number. The effective fermions then form an integer quantum Hall state with filling factor $\nu = n$ [10]. The corresponding bosonic state is a fractional quantum Hall liquid with filling factor $\nu = n/(1+n)$. This fractional liquid is a multicomponent state made out of n copies of a $\frac{1}{1+n}$–Laughlin–liquid [11], one for each internal state. Thus, by increasing the number of internal states we can go beyond $\nu = 1/2$, and lead the system into new fractions in the sequence $\nu = 1/2, 2/3, 3/4, \ldots$. All type of anyons [12] with statistics p/q, $p < q$ can now be created by choosing the adequate spin degeneracy and inserting lasers with a polarization and frequencies which couple to a given atomic internal state (spin component).

An exotic situation appears when the number of internal states equals the number of atoms in the trap. The effective fermions form in this case a pure spinor state with no spatial dependence. The corresponding bosonic state is then the product of a completely antisymmetric spin state and a $\nu = 1$ Laughlin state, which is fully antisymmetric too. This means that as long as the spin degrees of freedom are not excited the behaviour of the system will be fermionic. We will show that the elementary excitations of the systems are true fermions with fermionic statistics, though the real components of the system are bosons.

The formalism of fermionization that we present in this work can be also applied to explore several interesting possibilities. A very attractive one consists of inducing pairing in the effective fermions, so that they form a superconducting state. We will show that this is in fact possible when the atomic interaction depends on the internal state. We will consider the case of atoms with two internal levels when the interaction between atoms in the same spin state is much stronger than the one between atoms in a different spin state. We will see that the effective fermions configurate in pairs, forming a paired state identical to the BCS state of electrons with spin triplet p–wave pairing. The corresponding bosonic state, a Pfaffian state, is very well known in the context of the quantum Hall effect [13].

This paper is organized as follows. In section II we develop the fermionization scheme that maps the problem of interacting bosons in a rapid rotating trap onto one of non-interacting fermions. Using this formalism we show in section III that, when the ratio N/n is an integer number, the ground state of the system is a $\nu = n/(1+n)$ fractional quantum Hall liquid. Starting with these correlated liquids, we show how to create anyons with $p/(1+n)$ fractional statistics, with $1 \leq p \leq n$. We also discuss a possible experiment to reveal directly the statistics of anyons in a Ramsey-type interferometer. In section IV we study the special situation in which $N = n$. In section V we illustrate the results of the preceeding sections with some numerical exact diagonalizations for $N = 4, 6, 8$ particles. The experimental conditions required to observe the fractional quantum Hall states and the anyons described in this work are analyzed in section VI. As an example of the interesting possibilities of the fermionization scheme we show in section VII how pairing in the effective fermions can be induced in a system of atoms with two internal levels. Finally, in section VIII we summarize the results of this work, and discuss a possible scenario to observe the fractional quantum Hall states and the anyons.

II. THE FERMIONIZATION

We consider a set of N bosonic atoms confined in a potential which rotates in the $x-y$ plane at a frequency Ω. The atoms have an internal degree of freedom, that we will call spin, which can take $n = 2s+1$ values. We will assume that the confinement in the z direction is sufficiently strong so that we can ignore the excitations in that direction, and we will consider that the resulting po-

tential in two dimensions is isotropic and harmonic. The Hamiltonian describing this situation in a frame rotating with the trap is:

$$H = \frac{1}{2}\sum_{i=1}^{N}\left(-\nabla_i^2 + r_i^2 - 2\frac{\Omega}{\omega}L_{iz}\right) + g\sum_{i<j}^{N}\delta(\mathbf{r_i}-\mathbf{r_j}), \tag{1}$$

with L_{iz} being the z component of the angular momentum of the i–th atom, and where we have used the trap energy, $\hbar\omega$, as the unit of energy, and $\ell = (\hbar/m\omega)^{1/2}$ as the unit of length. The atoms are interacting via an effective contact potential, with an interacting coupling constant g related to the s–wave scattering length, a, and to the localization length in the z direction, ℓ_z, by $g = \sqrt{2/\pi}a/\ell_z$. In most parts of this paper we will also assume that the interaction between the atoms is spin independent.

Following our previous work [9] we begin by writing the Hamiltonian (1) in the following form:

$$H = H_B + H_L + V. \tag{2}$$

Here, $H_B = \sum_{i=1}^{N} -\nabla_i^2/2 + r_i^2/2 - L_{iz}$ is the quantum Hall single particle Hamiltonian, whose single particle energy levels are the Landau levels equally spaced by the cyclotron energy $2\hbar\omega$. The Hamiltonian $H_L = (1-\Omega/\omega)L_z$ is proportional to the z component of the total angular momentum, $L_z = \sum_{i=1}^{N} L_{iz}$, and V is the interaction term.

From now on we consider the *rapid rotation limit*, in which the energy scales characterizing Hamiltonians H_B and V are much larger than the one corresponding to H_L. In this limit the ground state and elementary excitations of the system must lie on the subspace $\mathcal{F}$ of common zero energy eigenstates of H_B and V.

We will show that, projected to the subspace $\mathcal{F}$, the bosonic problem described above exactly maps onto a problem of non-interacting fermions with n internal levels. We will first show that any wave function in $\mathcal{F}$ is the product of a $\nu = 1$ Laughlin state and a completely antisymmetric wave function in the lowest Landau level. We will then see that this antisymmetric wave function describes an effective system of non-interacting fermions.

Let Ψ be a wave function in $\mathcal{F}$. In order to be an eigenstate of H_B with eigenvalue zero, Ψ must lie within the subspace spanned by tensor products of single particle states in the lowest Landau level [10]. In the symmetric gauge these single particle states are labeled by the third component of the angular momentum m and the spin σ, and have the following form:

$$\varphi_{m\sigma}(\xi) = \frac{1}{\sqrt{\pi 2^{m+1}}} z^m e^{-|z|^2/4} \quad u^\sigma, \tag{3}$$

where $\xi = (z, u)$, $z = x + iy$, and u^σ is a s–spinor with components $u^\sigma(\beta) = \delta_{\sigma\beta}$, for $\sigma, \beta = -s, \ldots, s$. It follows that Ψ must have the form:

$$\Psi[\xi] = \sum_{\substack{m_1,\ldots,m_N\\ \sigma_1,\ldots,\sigma_N}} \alpha_{m_1,\ldots,m_N}\, z_1^{m_1}\ldots z_N^{m_N}\; u_1^{\sigma_1}\ldots u_N^{\sigma_N}, \tag{4}$$

with $m_i = 0, \ldots, \infty$, $\sigma_i = -s, \ldots, s$, and where for simplicity we have omitted the global exponential factor $\prod_k e^{-|z_k|^2/4}$[14].

The wave function Ψ must be an eigenstate of V with eigenvalue zero as well. We will show that the function $\prod_{i<j}(z_i - z_j)$ is a factor of Ψ. Let's choose any pair of particles i and j. The dependence of Ψ on z_i and z_j can be reexpressed in terms of the relative and center mass coordinates, z_{ij}, Z_{ij}, so that we can expand $\Psi[\xi] = \sum_m z_{ij}^m F_m$, where F_m depends on Z_{ij}, u_i, u_j, and on the positions and spinors of all the other particles. In order for Ψ to be annihilated by the hard-core interaction V, F_0 must be identically zero. It follows that $\forall i, j$, z_{ij} is a factor of Ψ, so that

$$\Psi[\xi] = \Phi[\xi]\prod_{i<j}(z_i - z_j), \tag{5}$$

where $\Phi[\xi]$ is an analytic function of z_i for all i, and therefore it lies in the lowest Landau level.

The factor $\Pi[z] = \prod_{i<j}(z_i - z_j)$ is proportional, up to a normalization constant, to the $\nu = 1$ Laughlin wave function. Since this factor is fully antisymmetric, the function Φ must be completely antisymmetric too, for Ψ to be a bosonic wave function. The factorization (5) thus states that any *bosonic* wave function lying within the subspace $\mathcal{F}$ is completely specified by a *fermionic* wave function in the lowest Landau level.

We will derive now an effective Hamiltonian for the fermionic wave function Φ, in the sense that the spectrum of this effective Hamiltonian will be identical to the one of Hamiltonian (2), projected onto the subspace $\mathcal{F}$.

Let $\Psi_n = \Phi_n\Pi$ be an eigenstate of Hamiltonian (2) with eigenvalue E_n. This state satisfies

$$H\Psi_n = H_L\Psi_n = E_n\Psi_n. \tag{6}$$

Since Π is an eigenfunction of H_L with eigenvalue $(1 - \Omega/\omega)N(N-1)/2$, we can write the above expression in the form:

$$\begin{aligned} E_n\Phi_n\Pi &= (H_L\Phi_n)\,\Pi + \Phi_n\,(H_L\Pi) && (7)\\ &= (1-\Omega/\omega)[(L_z + N(N-1)/2)\,\Phi_n]\Pi. && (8)\end{aligned}$$

We have then proved that if Ψ_n is an eigenstate of Hamiltonian (2) the corresponding fermionic state Φ_n is an eigenstate of the effective Hamiltonian

$$H_{eff} = (1-\Omega/\omega)\,[L_z + N(N-1)/2]\,, \tag{9}$$

with the same eigenvalue. This means that our problem of N interacting bosons projected to the subspace $\mathcal{F}$ reduces to an effective problem of N *free* fermions in the lowest landau level, governed by the Hamiltonian

(9). The spectrum of the real bosonic system is identical to the one of the effective free fermions, with the same eigenvalues and the same density of states. The fermionic eigenstates are connected to the bosonic ones by the transformation (5). Note that this transformation is not unitary, and therefore an orthonormal set of fermionic states will be transform into a set of bosonic states that will not be, in general, orthonormal.

The problem of free fermions in the lowest Landau level with Hamiltonian (9) can be easily solved. The ground state is a Fermi sea, in which the effective fermions fill up the single particle states $\varphi_{m\sigma}$ in order of increasing angular momentum m. Note that for each angular momentum m we have n single particle degenerate states. For a given number of particles N we can write $N = kn + r$, with $0 \leq r < n$. This means that we will have the first k single particle levels completely occupied and the $k+1$ level partially occupied with r fermions. The ground state will have degeneracy $\begin{pmatrix} n \\ r \end{pmatrix}$, corresponding to the different choices for the internal states of the last r fermions.

If Φ_{FS} is the wave function corresponding to the Fermi sea, the bosonic ground state is then just:

$$\Psi[\xi] = \Phi_{FS}[\xi] \prod_{i<j} (z_i - z_j). \tag{10}$$

Within this effective fermionic picture it is also very simple to find the low-lying excitations of the system. They are fermion-like excitations in which an effective fermion crosses the Fermi level, leaving a fermionic–hole in the Fermi sea.

III. MULTICOMPONENT ATOMIC FRACTIONAL QUANTUM HALL LIQUIDS AND THEIR ANYONS

In this section we will study the case in which the ratio N/n is an integer number k, so that we have a non-degenerate ground state with k completely occupied fermionic levels. We will show that the effective fermions form integer quantum Hall states at filling factor $\nu = n$. The corresponding bosonic states are $\nu = n/(1+n)$ fractional quantum Hall liquids with $1/n$–anyons.

A. The $\nu = 1/2$ Laughlin liquid

We start out with the trivial case in which $n = 1$ so that there is no spin degeneracy. The effective fermions will in this case occupy the single particle states with angular momentum $m = 0, \ldots, N-1$. The Slater determinant corresponding to this situation has the simple form:

$$\Phi_1[z] = \prod_{i<j}^{N} (z_i - z_j). \tag{11}$$

This state is the $\nu = 1$ Laughlin state, the same as for electrons in the quantum Hall effect. The density of this state is nearly uniform, with one fermion on average per unit area. The corresponding bosonic state,

$$\Psi[z] = \prod_{i<j}^{N} (z_i - z_j)^2, \tag{12}$$

is the $\nu = 1/2$ Laughlin liquid that we studied in our previous work [9]. Starting from this liquid one can create 1/2–atomic–quasihole excitations with 1/2–fractional statistics. The fractional statistical phase of these quasiholes may be revealed directly in a Ramsey-type interferometer [9].

B. The $\nu = 2/3$ liquid

Before considering the general case of any number of internal states, it is illuminating to study the case of two internal states (and an even number of atoms). We will use the spin language, and say that we have atoms with spin $s = 1/2$ that can be either up ($\uparrow$) or down ($\downarrow$).

The effective fermions will fill up the single particle states with angular momentum $m = 0, \ldots, N/2 - 1$ and both spin states $\sigma = \uparrow, \downarrow$. We will have $N/2$ fermions with spin up and $N/2$ fermions with spin down, and each group will form a $\nu = 1$ Laughlin state. The wave function corresponding to this situation is

$$\Phi[\xi] = \mathcal{A} \left\{ \phi_1^{\uparrow}(\xi_1, \ldots, \xi_{N/2}) \times \phi_1^{\downarrow}(\xi_{N/2+1}, \ldots, \xi_N) \right\}, \tag{13}$$

where

$$\phi_1^{\sigma}[\xi] = \prod_{i<j}^{N/2} (z_i - z_j) u_i^{\sigma} u_j^{\sigma}, \qquad \sigma = \uparrow, \downarrow, \tag{14}$$

is a $\nu = 1$ Laughlin state of $N/2$ fermions in the spin state σ, and the operator $\mathcal{A}$ antisymmetrizes over all possible ways of distributing the fermions in two groups of $N/2$ fermions. The state (13) is a $\nu = 2$ quantum Hall state, made out of two $\nu = 1$ states, one with spin up and the other with spin down. The density profile of this state is nearly flat in the bulk, with two fermions per unit area on average, one with spin up and the other with spin down.

The corresponding bosonic state is obtained from the wave function (13) multiplying by the factor $\prod^{N} (z_i - z_j)$. To gain insight about the properties of this bosonic state it is very convenient to use the language of the quantum Hall effect, that relates angular momenta, filling factors and densities [10]. In general, given an incompressible liquid in the lowest Landau level, its filling factor ν is related to its total angular momentum by $L \sim N^2/2\nu$. Furthermore, the liquid has a uniform density in the bulk, with ν particles per unit area on the average.

Multiplication by the factor Π increases the total angular momentum L_F of the original fermionic state, so that the resulting bosonic state has a total angular momentum

$$L = L_F + \frac{N(N-1)}{2}. \quad (15)$$

This means that the new bosonic state will have a filling factor

$$\nu = \frac{\nu_F}{1+\nu_F}. \quad (16)$$

As $\nu_F = 2$ we have that the bosonic state is a $\nu = 2/3$ fractional Hall liquid. This fractional liquid is made out of two $\nu = 1/3$ liquids, one with spin up and the other with spin down. The average density in this state is of $2/3$ atoms per unit area, $1/3$ with spin up, and $1/3$ with spin down. Roughly speaking, we can say that one bosonic atom is made out of three effective fermions, or that one fermion corresponds to $1/3$ atom. This counting will be very useful to study the anyonic excitations in the fermionic picture.

C. The $\nu = n/(1+n)$ multispinor liquids

After the above discussion the generalization to any number of internal states is straightforward. The effective fermions will fill up the single particle states with angular momenta $m = 0, \ldots, N/n - 1$, and spin $\sigma = -s, \ldots, s$. We will have n groups of N/n fermions, one for each spin state, and each group will form a $\nu = 1$ polarized liquid of the form

$$\phi_1^\sigma[\xi] = \prod_{i<j}^{N/n} (z_i - z_j) u_i^\sigma u_j^\sigma. \quad (17)$$

The fermionic ground state is the product of the wave functions of the different groups, antisymmetrized over all possible ways of distributing N particles in groups of N/n particles:

$$\Phi[\xi] = \mathcal{A}\left\{\prod_\sigma \phi_1^\sigma[\xi]\right\} \quad (18)$$

The state (18) is a $\nu = n$ quantum Hall state, consisting of n polarized $\nu = 1$ liquids, one for each possible spin state. In this state we have n fermions per unit area, one in each of the spin states.

The relation (16) tells us that the corresponding bosonic state will be a fractional state with filling factor $\nu = n/(1+n)$. This liquid is made out of n polarized $\nu_\sigma = 1/(1+n)$ fractional liquids, one for each spin state σ. We will have $n/(1+n)$ atoms per unit area, $1/(1+n)$ per each spin state.

By increasing the spin degeneracy $n = 1, 2, 3, \ldots$, one can drive the atomic system into fractional quantum Hall states in the sequence $\nu = 1/2, 2/3, 3/4, \ldots$.

D. Anyons

We will show now how to create anyons starting from the fractional atomic quantum liquids presented in the preceeding subsection. Following the ideas of our previous work [9] we insert a laser localized (within an area ℓ^2) at some position η. We require $|\eta|$ to be much smaller than the size of the quantum Hall state ($\sim \nu^{-1}\sqrt{N-1}$), so that the border effects are negligible. We will also assume that the laser only couples to atoms with spin σ_0. The presence of the laser can be described by a localized repulsive potential, so that the new Hamiltonian for the bosonic atoms can be approximated by:

$$H^{\eta\sigma_0} = H + V_0 \sum_i \delta(z_i - \eta) P_i^{\sigma_0}, \quad (19)$$

where the operator $P_i^{\sigma_0}$ projects the spinor of the i–th particle on the state with spinor $u_i^{\sigma_0}$. In the same way as we did in the previous section we can project Hamiltonian (19) onto the subspace $\mathcal{F}$ of wave functions of the form $\Psi = \Phi \prod_{i<j}(z_i - z_j)$, and derive an effective Hamiltonian for the fermionic wave function Φ, which has now the form [15]:

$$H_{eff}^{\eta\sigma_0} = H_{eff} + V_0 \sum_i \delta(z_i - \eta) P_i^{\sigma_0}. \quad (20)$$

We will find the ground state of the fermionic Hamiltonian (20) and show that the corresponding bosonic state is a fractional quasihole state localized at position η. We consider the limit in which the intensity of the laser is much larger that the energy scale characterizing Hamiltonian H_{eff}. In this limit the new ground state $\Phi_{\eta\sigma_0}$ must be a zero eigenstate of the potential created by the laser. It follows that, whenever the i–th particle is in the spin state σ_0, $(z_i - \eta)$ must be a factor of $\Phi_{\eta\sigma_0}$.

All groups with $\sigma \neq \sigma_0$ will remain in a $\nu = 1$ Laughlin state with wave function $\phi^\sigma[z]$. But the group with $\sigma = \sigma_0$ will form the quasihole state

$$\phi_\eta^{\sigma_0}[\xi] = \prod_{i=1}^{N/s} (z_i - \eta) \phi^{\sigma_0}[\xi]. \quad (21)$$

The resulting state is a $\nu_F = n$ fermionic liquid in which one fermion with spin σ_0 has been removed at position η. As each boson is made out of $n+1$ fermions, the corresponding bosonic state will be a $\nu = n/(1+n)$ liquid in which a fraction $1/(1+n)$ of an atom is missing in the component with spin σ_0. As it follows from the theory of fractional quantum Hall effect, these atomic fractional quasiholes are anyons with $1/(1+n)$ fractional statistics.

In the analysis above we have considered a situation in which the laser only couples to atoms in a particular internal state σ_0. But we can also imagine situations in which the laser coupling affects different internal states. By selecting the number of internal states coupled to the laser we can create anyons with any fractional statistics of the form $p/(1+n)$, with $1 \leq p \leq n$.

The fractional statistics of the anyons we have described may be directly proved in an experiment similar to the one we proposed for detecting $\frac{1}{2}$–anyons [9]. Suppose that we want to detect $\frac{1}{3}$–anyons. Then we will first prepare a system of bosons with 2 internal states, up and down, in a $\nu = 2/3$ state. We focus then a laser, that only couples to atoms with spin up (or down), at position η_1, and increase its intensity until a single $\frac{1}{3}$–quasihole is created. Keeping constant the intensity of this laser we then adiabatically insert another laser at position η_2, far enough from η_1, affecting also only atoms with spin up (or down). The system then evolves from the one–quasihole state, Ψ_{η_1}, to a two–quasihole state, Ψ_{η_1,η_2}. The crucial point is that at a certain point of the evolution the system reaches a superposition state $\Psi \sim \Psi_{\eta_1} + \Psi_{\eta_1,\eta_2}$. This is the superposition we need to test the statistical angle. If we adiabatically move the laser at position η_1 along a path enclosing position η_2, the evolved state at the end of the process will be

$$\Psi' \sim \Psi_{\eta_1} + e^{i2\pi/3}\Psi_{\eta_1,\eta_2}, \tag{22}$$

where the relative phase, $e^{i2\pi/3}$, reflects the statistical phase. This phase may be detected in a Ramsey-type interferometer in which the evolution of the initial superposition state, Ψ, is split in three different ways, in which the laser at η_1 performs respectively, one, two, and three loops enclosing η_2. If after performing the loops we continue the evolution of the system, one can check that the system will only evolve to the two-quasihole state, Ψ_{η_1,η_2}, in the case in which 3 loops were performed. This 3 reflects the $\frac{1}{3}$–statistics of the quasiparticles.

IV. THE FERMIONIC BOSONS

In this section we give a special attention to the case in which the degeneracy of spin equals the number of atoms in the system. Such a situation is a realistic possibility with a system of small number of atoms. In this case the effective fermions will occupy the single particle states with $m = 0$ and spin $\sigma = 1, \ldots, N$. The corresponding Slater determinant is a pure spinor function, with no spatial dependence

$$\Phi[\xi] = \chi[u] = \sum_{i_1,\ldots,i_N} \epsilon_{i_1,\ldots,i_N} u^1_{i_1} \ldots u^N_{i_N}, \tag{23}$$

so that the bosonic ground state

$$\Psi[\xi] = \chi[u] \prod_{i<j}(z_i - z_j) \tag{24}$$

is the product of a completely antisymmetric spin state and a $\nu = 1$ Laughlin state. We note that in the above wave function the spin and orbital degrees of freedom are completely decoupled . The orbital wave function is antisymmetric, so that as long as the spin degrees of freedom are not excited the behaviour of the system will be fermionic. Any property concerning orbital degrees of freedom that were experimentally measured would behave as if we would have fermions in the system, even though our system is composed of bosons.

Suppose, for example, that we insert a laser localized at a certain position η, and that the laser couples to all atoms, no matter the internal state. As the intensity of the laser is increased the system will evolve from state (24) to the quasihole state:

$$\Psi_\eta[\xi] = \chi[u] \prod_i (z_i - \eta) \prod_{i<j}(z_i - z_j). \tag{25}$$

This quasihole state is a $\nu = 1$ state in which one particle has been removed at position η, therefore, the relation between filling factors and fractional statistics states that we have created an anyon with statistics 1; that is, a fermion.

V. NUMERICAL SIMULATIONS

In this section we illustrate the results presented in the previous sections with some numerical exact diagonalizations for $N = 4, 6$, and 8 bosons. In our calculations we restrict ourselves to single-particle states in the lowest Landau level. This approximation is valid in the limit in which the energy separation between Landau levels, 2ω, is much larger than all the energies avaliable.

We note that Hamiltonian (2) is invariant under rotations around the z axis, and also under global spin rotations. These symmetries allow us to diagonalize it within subspaces of fixed z component of the total angular momentum, L, fixed total spin, S, and fixed z component of the spin, S_z.

Figure 1 shows the eigenvalues of Hamiltonian (2) for a system of $N = 6$ bosons with two internal levels, in the rapid rotation limit. We see that there is a branch of states well separated by a gap from the rest of the spectrum. As long as the temperature is smaller than this gap, the system will remain within this subspace. All the states in this subspace lie within the lowest Landau level and have zero interaction energy. They form what we have called subspace $\mathcal{F}$.

In order to check the fermionization scheme we have diagonalized the effective Hamiltonian (9) for a system of $N = 6$ fermions with spin $s = 1/2$. Figure 2 shows that the bosonic spectrum projected to the subspace $\mathcal{F}$ is identical to that of the free fermions, except for a shift in the angular momentum of the eigenstates, $\Delta L = N(N-1)/2$. This is precisely the angular momentum of the $\nu = 1$ Laughlin state, that connects fermionic and bosonic states. For a given angular momentum, there are many degenerate states within the subspace $\mathcal{F}$. This density of states is exactly reproduced by the system of non-interacting fermions (Figure 3), exept again, for the shift in angular momentum.

We have studied the ground state properties of a system of $N = 4, 6, 8$ bosons, and n internal states, for the

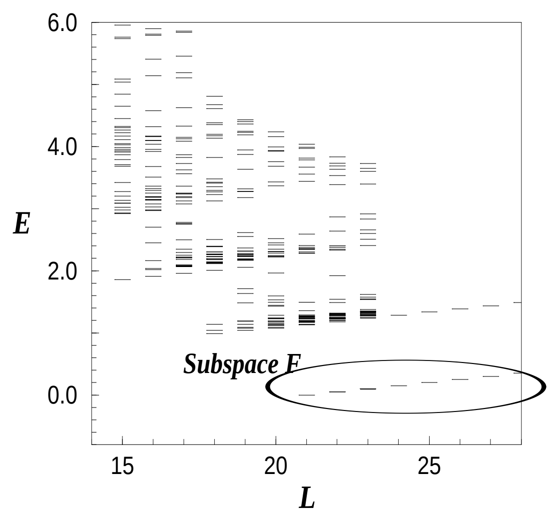

FIG. 1: Eigenvalues of Hamiltonian(2) for $N = 6$ bosons and $n = 2$ internal levels, as a function of the total angular momentum L. For illustration purposes we chose $(1 - \Omega/\omega)/g = 0.01$. The energy is measured in units of $g\hbar\omega$, and all energies are shifted by $\Delta E = -2.1 g\hbar\omega$. For total angular momenta larger than $L = 23$ the dimension of the subspace is too large and we can not diagonalize the Hamiltonian exactly. We have used a variational approach to obtain the ground state and the first excited states.

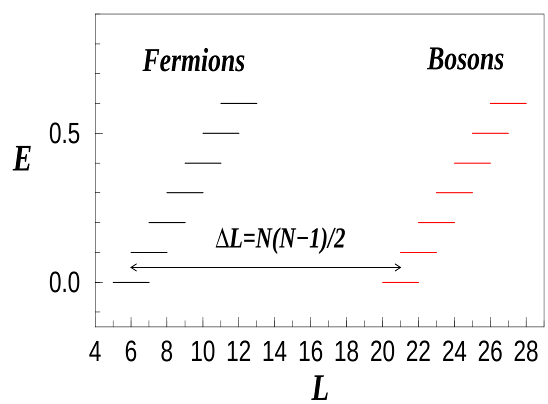

FIG. 2: Eigenvalues of Hamiltonian (2) for $N = 6$ bosons projected to the subspace $\mathcal{F}$ (right), and eigenvalues of Hamiltonian (9) for $N = 6$ fermions (left), as a function of the total angular momentum L. Both spectra are identical except for a shift in the angular momentum $\Delta L = N(N-1)/2$. The data shown correspond to $(1 - \Omega/\omega)/g = 0.01$. The energy is measured in units of $g\hbar\omega$, and all energies are shifted by $\Delta E = -2.1 g\hbar\omega$.

cases in which N/n is an integer number. We find (Figure 4) that the total angular momentum L_0 of the ground state is always equal to the one of the Fermi sea formed by the fermions, plus the angular momentum of the $\nu = 1$

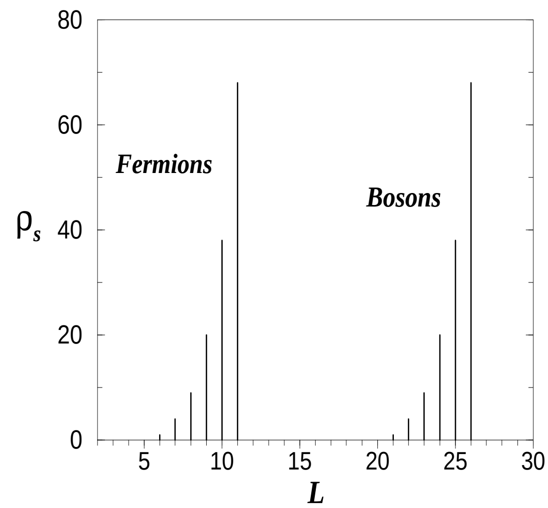

FIG. 3: Density of states for $N = 6$ bosons with $n = 2$ internal levels in the subspace $\mathcal{F}$, as a function of the total angular momentum L. We have also plotted the density of states for a system of $N = 6$ fermions and $n = 2$ internal levels with Hamiltonian (9).

Laughlin state:

$$L_0 = N(N/n - 1) + N(N-1)/2. \tag{26}$$

We have checked that the effective fermions form integer quantum Hall states at filling factor $\nu = n$. The density profile of these states is nearly flat in the bulk, with n fermions per unit of area (Figure 5). The corresponding bosonic states are fractional quantum Hall liquids at filling factor $\nu = n/(n+1)$. The density profile of these states is also nearly flat in the bulk, with $n/(1+n)$ bosons per unit area (Figure 6).

We have also checked that, starting from these fractional liquids, anyons with $p/(1+n)$ statistics may be created. We consider first the following Hamiltonian,

$$H_0 = H + V_1 \sum_i \delta(z_i - \eta_1) P_i^{\sigma_1 \dots \sigma_p}, \tag{27}$$

which includes the presence of a laser localized at position η_1, affecting p different spin states. We have diagonalized Hamiltonian (27) within the subspace $\mathcal{F}$ for $N = 6$ bosons, $n = 1, 2, 3, 6$, and $1 \le p \le n$. When the laser power is sufficiently large we find that the ground state is a one-quasihole state, Ψ_{η_1}, in which a certain amount of atoms is missing in the components with spin states $\sigma_1, \dots, \sigma_p$. We have also diagonalized a Hamiltonian describing the system in the presence of two lasers localized at positions η_1 and η_2, coupled to the same p internal states. The resulting ground state, Ψ_{η_1,η_2}, consists of two identical quasiholes located at positions η_1 and η_2. Having the states Ψ_{η_1} and Ψ_{η_1,η_2} we can check very easily what is the statistics of the quasiholes we have created. Suppose that we adiabatically drive the laser at

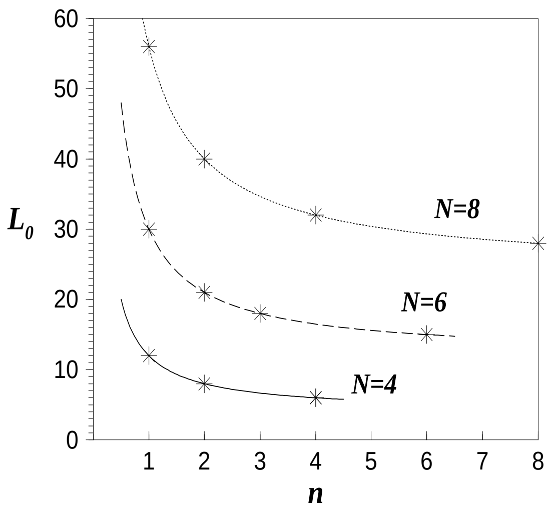

FIG. 4: Total angular momentum L_0 of the ground state for a system with $N = 4, 6, 8$ bosons, as a function of the number of internal levels n. We have plotted the numerical values (stars) together with the curve given by equation (26).

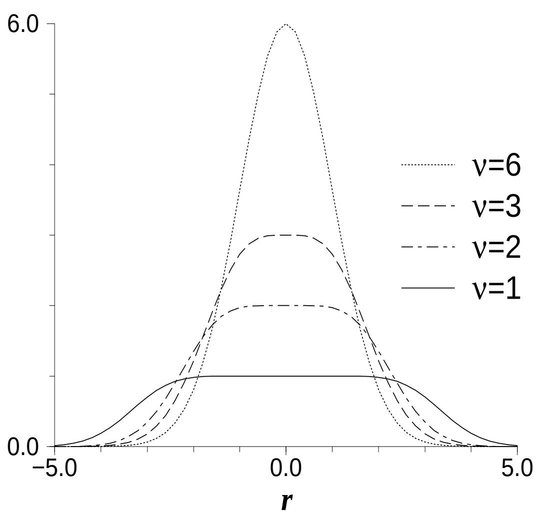

FIG. 5: Density profile of the ground state of a system of $N = 6$ fermions with Hamiltonian (9). The different curves correspond to different values of the number of internal states n. The unit of density is $1/(2\pi\ell^2)$, and the unit of length is ℓ.

position η_1 along a path enclosing position η_2. At the end of the process the states Ψ_{η_1} and Ψ_{η_1,η_2} will pick up Berry phases given, respectively, by,

$$\begin{aligned} \gamma_1 &= 2\pi \int_A dxdy |\Psi_{\eta_1}|^2 \\ \gamma_2 &= 2\pi \int_A dxdy |\Psi_{\eta_1,\eta_2}|^2, \end{aligned} \tag{28}$$

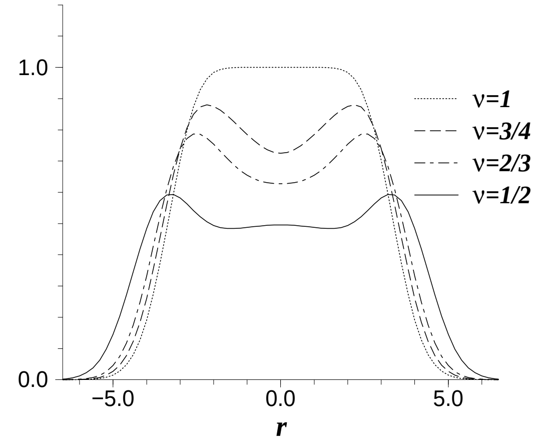

FIG. 6: Density profile of the ground state of a system of $N = 6$ bosons with Hamiltonian (2), in the limit of rapid rotation. The different curves correspond to different values of the number of internal states n. The unit of density is $1/(2\pi\ell^2)$, and the unit of length is ℓ.

where the integrals are performed over the area A enclosed by the path described by the laser. The equations (28) for the Berry phases can be easily derived, taking into account that the states Ψ_{η_1} and Ψ_{η_1,η_2} are made out of fractional Laughlin quasihole states [12]. The difference between the Berry phases $\gamma_2 - \gamma_1$ will reflect the extra phase that the quasihole at η_1 picks up because of the presence of the other quasihole at η_2. Since the closed loop we have performed is equivalent to two consecutive interchanges of the quasiholes we have that the statistics of the quasiparticles is given by the angle

$$\theta = \frac{\gamma_2 - \gamma_1}{2}. \tag{29}$$

We have calculated expression (29) for quasihole states corresponding to $N = 6$ bosons and different numbers of internal states $n = 1, 2, 3, 6$. The statistical phases we obtain (Figure 7) are in excellent agreement with the fractions $p/(1 + n)$, even though the finite size of the system.

VI. EXPERIMENTAL CONDITIONS

In this section we discuss the set of conditions that a system of bosonic atoms must fulfill in order to observe the fractional quantum Hall liquids and the anyons we have described in the preceeding sections.

First of all we need a system with degenerate internal levels, such that the interaction between the atoms does not depend on the internal state. We have also made a two-dimensional approximation, so that the atoms must be confined in a trap with $\ell \ll \ell_z$. For the ground state to be a fractional quantum Hall liquid the system must

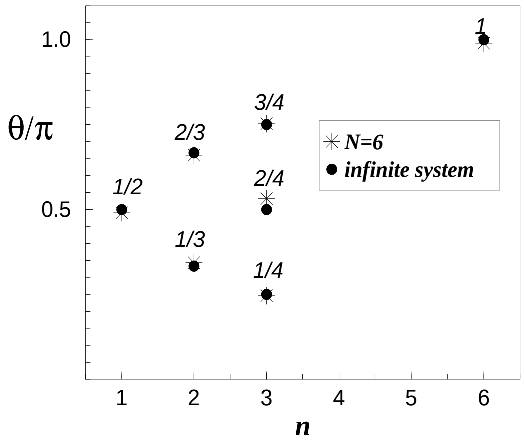

FIG. 7: Statistical angle of the quasiholes created in a system of $N = 6$ bosons, as a function of the number of internal states. We have plotted the numerical results (stars) together with the ones predicted for an infinite system (dots). The different points for each number of internal states correspond to different values of the number of internal states the laser is coupled to.

be in the rapid rotation regime. From the numerical diagonalizations we can calculate the critical frequency of rotation at which the system will be driven to a correlated quantum Hall liquid. For a system of $N = 6$ particles we find that the ground state of the system becomes a $\nu = 1/2, 2/3, 3/4$ liquid for a frequency of rotation $\Omega/\omega \geq 0.80, 0.62, 0.38$, respectively. These data show how the critical frequency of rotation decreases as the number of internal states is increased, so that the observation of the fractional states $\nu = n/(1+n)$ requires less faster rotating traps, as we approximate $\nu = 1$. For an arbitrary number of particles N we can estimate what is the condition that the frequency of rotation must fulfill in order for the a state $\nu = n/(1+n)$ to be the ground state. The state $\nu = n/(1+n)$ has a total angular momentum $L \sim \nu^{-1}N(N-1)$ (decreasing as $\nu \to 1$), and thus an angular momentum energy per particle $e_L \lesssim \nu^{-1}(1-\Omega/\omega)(N-1)$. In order to be in the rapid rotation regime we need e_L to be much smaller than both the trap energy and the typical interaction energy, so that the conditions $(1-\Omega/\omega)(N-1) \ll \nu,\ \nu g/4\pi$ must be fulfill.

For creating the anyons the conditions required are much more demanding. We have first to focus the lasers within a distance $\sim \ell$. For a localization length $\ell \sim 1\mu m$ this implies an upper limit for the trap frequency of $\sim 1kHz$. As well, the frequency of rotation needed to create the anyonic excitations is larger that the critical frequency to observe the fractional quantum Hall liquids, since the total angular momentum of the quasihole states is $2N$ units of angular momentum larger. For a system with $N = 6$ particles anyons with $1/2, 1/3, 1/4$–statistics may be created for $\Omega/\omega \geq 0.91, 0.81, 0.68$, respectively.

Finally , the most restrictive condition to observe the fractional quantum Hall states and their anyons is the temperature. In order to freeze out the excitations we need $kT/\hbar\omega \ll (1-\Omega/\omega)$. For a system of $N = 6$ particles the observation of the states $\nu = 1/2, 3/4, 3/4$ requires temperatures $kT/\hbar\omega \ll 0.20, 0.38, 0.62$.

VII. THE PAIRED SUPERCONDUCTING STATES

In this section we study a situation in which the ground state of the atomic system is a paired state of effective fermions. We consider a system with and even number of atoms and $n = 2$ internal levels, up ($\uparrow$), and down ($\downarrow$), and we assume that the interaction between atoms in the same spin state is much larger than the interaction between atoms with different spin states. In this case the interaction term in Hamiltonian (2) can be approximated by

$$V^{\parallel} = g\sum_{i<j}^{N}\delta(z_i - z_j)\left(P_{ij}^{\uparrow\uparrow} + P_{ij}^{\downarrow\downarrow}\right), \qquad (30)$$

where the operator $P_{ij}^{\sigma\sigma}$ projects the spin state of the pair of particles ij on the state $\sigma\sigma$.

We will show that, in the limit of rapid rotation, the ground state of the system is the product of a $\nu = 1$ Laughlin state and a completely antisymmetric wave function. This antisymmetric wave function has the structure of a BCS state of fermions with p-wave pairing.

From the results of section II it follows that in the rapid rotation limit the ground state, Ψ, of the system must be both an analytic function of the z_i's and an eigenstate of the interaction (30) with eigenvalue 0. Let's choose any pair of particles i and j. For Ψ to be annihilated by the interaction (30) we have two possibilities. Either the particles i and j are in a different spin state, or the relative angular momentum of the pair is larger than 0. It follows that either the spin triplet $(\uparrow_i\downarrow_j + \downarrow_i\uparrow_j)$ or the factor $(z_i - z_j)$ is a factor of Ψ. How many factors of the form $(z_i - z_j)$ and how many spin triplets we will have in the ground state will be the choice that costs the minimal amount of angular momentum energy, thus, we will have as many triplet states as possible. Note that the maximum number of pairs that we can arrange in triplets is precisely $N/2$, corresponding to a certain pairing of the particles. It follows that the ground state has the form

$$\Psi = Pf\left(\frac{\uparrow_i\downarrow_j + \downarrow_i\uparrow_j}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j), \qquad (31)$$

where the Pfaffian Pf is defined by

$$Pf(M_{ij}) = \mathcal{A}(M_{12}M_{34}\ldots M_{N-1,N}), \qquad (32)$$

where M_{ij} are the elements of an antisymmetric matrix and $\mathcal{A}$ denotes the operation of antisymmetrization, normalized such that each distinct term appears one with coefficient 1.

The state (31) is the product of a $\nu = 1$ Laughlin state, in which the factor $(z_i - z_j)$ appears for all pairs of particles ij, and a Pfaffian factor. The Pfaffian factor acts in the following way. It chooses a pairing of the particles, and for all pairs ij within the pairing it removes the factor $(z_i - z_j)$ and substitutes it by a triplet state $(\uparrow_i\downarrow_j + \downarrow_i\uparrow_j)$. In this way the system minimizes its total angular momentum, remaining in the subspace of zero interaction eigenstates.

Since the Pfaffian factor is completely antisymmetric we can interpret it as the wave function of a system of N effective fermions, in the same way that we did in section II. In this case, however, the fermionic wave function is not an analytic function of the z_i's, and, therefore it does not lie within the lowest Landau level. Such a Pfaffian state has been shown to be the ground state solution of a BCS theory for fermions with p-wave triplet pairing [13]. With this result we can say that the state (31) is a superconducting state of effective fermions. It would very interesting to derive an effective Hamiltonian for the fermionic wave function, in a similar way as we did in section II. This effective Hamiltonian should describe in this case fermions with attractive interactions between particles with different spin states. Derivation of this Hamiltonian will be discussed elsewhere.

VIII. CONCLUSIONS

In conclusion, we have shown how by rapidly rotating a gas of ultra cold bosonic atoms, we can drive the system into a strong interacting regime in which novel correlated ground states and excitations appear.

We have developed a theory that exactly maps the problem of interacting bosons in the rapid rotation regime to a problem of non–interacting fermions. This fermionization scheme allows us to find the exact ground state and low-lying excitations of the system. For the case of a system with no internal levels we recover the results of our previous work, with a $\frac{1}{2}$–Laughlin liquid as the ground state, and $1/2$ anyonic excitations. In the presence of n internal levels and when the ratio N/n is an integer number, the ground state of the effective fermions is a $\nu = n$ integer quantum Hall state. The corresponding bosonic ground state is a fractional multicomponent quantum Hall liquid with filling factor $\nu = n/(1 + n)$. This fractional liquid is a highly correlated state made out of n copies of a $1/(1 + n)$–Laughlin liquid, one for each internal state. Starting from these liquids we have shown how to create $p/(1+n)$–anyons, by focusing lasers at the desired positions, coupled to p different spin states. The fractional statistics of these anyons may be directly tested in a Ramsey-type interferometer similar to the one we proposed to detect $1/2$ anyons.

The experimental observation of the fractional quantum Hall liquids and the anyons described in this work requires small temperatures, small number of atoms, and very fast rotating traps. The conditions required are relaxed as the number of internal levels increases. A possible scenario for the observation of these entangled states is a two–dimensional rotating optical lattice. As already shown in the recent experiment with a one-dimensional optical lattice it is possible to reach a regime in which a small fixed number of atoms is confined in each well of the lattice. Furthermore, since one has many identical copies of the system, the experimental signals are highly magnified.

Acknowledgments

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[14] For the sake of short notation, in most part of this paper we omit the exponential factor $\prod_k e^{-|z_k|^2/4}$, as well as the normalization factors.
[15] Strictly speaking one obtains the Hamiltonian (20) by defining a new scalar product in the fermionic Hilbert space of the form $\langle \Phi_1 | \Phi_2 \rangle = \int d\mu^2 \Phi_1^* \Phi_2$, where $d\mu^2 = \prod_{i<j} |z_i - z_j|^2 dxdy$.